(c) $0.5\overline{5}$

SOLUTION: $0.5\overline{5} = 0.5 + 0.05 + 0.005 + 0.0005 + \ldots$

Thus $a = 0.5$ and $r = \frac{0.05}{0.5} = \frac{5}{50} = \frac{1}{10}$

Hence, $\lim_{n\to\infty} S_n = \frac{\frac{1}{2}}{1-\frac{1}{10}} = \frac{\frac{1}{2}}{\frac{9}{10}} = \frac{5}{9}$

(d) $0.3\overline{6}$

SOLUTION: What are the details of the required thought-processes?

$$
\begin{aligned}
10N &= 3.6\overline{6} \\
-\quad N &= 0.3\overline{6} \\
\hline
9N &= 3.30 \\
N &= \frac{3.3}{9} = \frac{33}{90} = \frac{11}{30}
\end{aligned}
$$

What is the first term? 0.3

What is the second term? 0.06

What is the third term? 0.006

$t_3 = 0.1 t_2$

but $t_2 \neq 0.1 t_1$

$$
\begin{aligned}
0.3\overline{6} &= 0.3 + 0.06 + 0.006 + 0.0006 + \ldots \\
&= 0.3 + 0.06 + 0.06(0.1) + 0.06(0.1)^2 + \ldots
\end{aligned}
$$

So, $0.3\overline{6} = 0.3 + \dfrac{0.06}{1-0.1}$

$$= 0.3 + \frac{0.06}{0.9} = \frac{3}{10} + \frac{6}{90} = \frac{3}{10} + \frac{1}{15}$$

$$= \frac{45+10}{150} = \frac{55}{150} = \frac{11}{30}$$

(e) $4.\overline{123}$

SOLUTION. There are two approaches we could take. For posterity, I will show both.

$$\begin{array}{rl} 1000N = & 4123.\overline{123} \\ -\quad N = & -\ 4.\overline{123} \\ \hline 999N = & 4119.0 \end{array}$$

$\therefore\ N = \dfrac{4119}{999} = \dfrac{1373}{333}$

$4119 \div 3 = 1373$

The approach using geometric series:

$$4.\overline{123} = 4 + 0.123 + 0.000123 + \ldots$$

$$= 4 + 0.123 + 0.123(0.001) + 0.123(0.001)^2 + 0.123(0.001)^3 + \ldots$$

$$= 4 + \frac{0.123}{1-0.001} = 4 + \frac{0.123}{0.999} = 4 + \frac{123}{999}$$

$123 \div 3 = 41$

$$= 4 + \frac{41}{333} = \frac{1332+41}{333} = \frac{1373}{333}$$

$333 \times 4 = 1332$

(f) $2.0\overline{31}$ First method:

SOLUTION.

$$1000N = 203.1\overline{31}$$
$$-\ \ N = 2.0\overline{31}$$
$$99N = 201.1$$

$$\therefore\ N = \frac{201.1}{99} = \frac{2011}{990}$$

Geometric Series Method:

$$2.0\overline{31} = 2 + 0.031 + 0.031(0.01) + 0.031(0.01)^2 + 0.031(0.01)^3 + \dots$$

$$= 2 + \frac{0.031}{1 - 0.01} = 2 + \frac{0.031}{0.99}$$

$$1800 + 180 = 1980$$

$$= 2 + \frac{31}{990} = \frac{1980 + 31}{990} = \frac{2011}{990}$$

(g) $6.7\overline{182}$ First method:

SOLUTION:

$$1000N = 6718.2\overline{182}$$
$$-\ \ N = 6.7\overline{182}$$
$$999N = 6711.50$$

$$\therefore\ N = \frac{6711.5}{999} = \frac{67115}{9990}$$

$$N = \frac{13423}{1998}$$

Sum of Geometric Series Method:

$$6.7\overline{182} = 6.7 + 0.0182 + 0.0182(0.001) + 0.0182(0.001)^2$$

$$= \frac{67}{10} + \frac{182}{10{,}000} + \frac{182}{10^4}\left(\frac{1}{10^3}\right) + \frac{182}{10^4}\left(\frac{1}{10^3}\right)^2 + \ldots$$

$$= \frac{67}{10} + \frac{\frac{182}{10^4}}{1 - \frac{1}{10^3}} = \frac{67}{10} + \frac{\frac{182}{10^4}}{\frac{999}{10^3}}$$

$$= \frac{67}{10} + \frac{182 \times 10^3}{999 \times 10^4} = \frac{67}{10} + \frac{182}{9990}$$

$$= \frac{67(999) + 182}{9990} = \frac{67115}{9990} = \frac{13423}{1998}$$

(h) $62.\overline{430716}$

SOLUTION: Using first method:

$$1{,}000{,}000\,N = 62430716.\overline{430716}$$
$$-\quad N = 62.\overline{430716}$$
$$999{,}999\,N = 62430654.0$$

$$\therefore N = \frac{62430654}{999{,}999} = \frac{145526}{2331}$$

On the following page I show the process using the defenition of the sum of a geometric series.

$$62.\overline{430716} = 62 + 0.430716 + 0.430716\left(\frac{1}{10^6}\right) + 0.430716\left(\frac{1}{10^6}\right)^2 + 0.430716\left(\frac{1}{10^6}\right)^3 + \ldots$$

$$= 62 + \frac{0.430716}{1 - \frac{1}{10^6}}$$

$$= 62 + \frac{0.430716}{\frac{999999}{10^6}}$$

$$= 62 + \frac{430716}{999999} = 62 + \frac{1004}{2331}$$

$$= \frac{62(2331) + 1004}{2331} = \frac{145526}{2331}$$

---

(6) The sum of the terms of an infinite geometric progression is 12 and the sum of the first two terms of the progression is 6.
Write the first three terms of the progression.

SOLUTION.

$S_n = 12 \quad , \quad S_2 = t_1 + t_2 = 6$

$6 = a + ar \longleftrightarrow r = \frac{6-a}{a}$

$t_3 = ar^2 = a\left(\frac{6-a}{a}\right)^2 = a\left[\frac{36-12a+a^2}{a^2}\right] = \frac{a^2-12a+36}{a}$

$S_n = 12 = \frac{a}{1-r} = \frac{a}{1-\left(\frac{6-a}{a}\right)} = \frac{a}{\frac{a-6+a}{a}}$

$12 = \frac{a^2}{2a-6} \longleftrightarrow 12(2a-6) = a^2$

$\longleftrightarrow a^2 - 24a + 72 = 0$

Completing the square:

$a^2 - 24a = -72 \longleftrightarrow a^2 - 24a + (-12)^2 = -72 + 144$

$\longleftrightarrow (a-12)^2 = 72 \longleftrightarrow a - 12 = \pm\sqrt{72}$

$\longleftrightarrow a - 12 = \pm 6\sqrt{2} \longleftrightarrow a = 12 \pm 6\sqrt{2}$

$\longleftrightarrow a = 12 + 6\sqrt{2}$ or $a = 12 - 6\sqrt{2}$

$\therefore$ There are two possible geometric progressions,

one with $r = \frac{6-(12+6\sqrt{2})}{12+6\sqrt{2}} = \frac{-6-6\sqrt{2}}{12+6\sqrt{2}}$

$= \frac{(-6-6\sqrt{2})(12-6\sqrt{2})}{(12+6\sqrt{2})(12-6\sqrt{2})} = \frac{-72+36\sqrt{2}-72\sqrt{2}+72}{144-72}$

$= \frac{-36\sqrt{2}}{72} = -\frac{\sqrt{2}}{2}$; $t_1 = 12 + 6\sqrt{2}$,

$t_2 = (12+6\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right) = -6\sqrt{2} - 6$; $t_3 = (12+6\sqrt{2})\left(-\frac{\sqrt{2}}{2}\right)^2$
$= 6 + 3\sqrt{2}$

99

So the first possible three terms are

$$12+6\sqrt{2},\ -6\sqrt{2}-6,\ 6+3\sqrt{2}$$

The other possible geometric progression has first term $12-6\sqrt{2}$, with

$$r = \frac{6-(12-6\sqrt{2})}{12-6\sqrt{2}} = \frac{-6+6\sqrt{2}}{12-6\sqrt{2}}$$

$$= \frac{(-6+6\sqrt{2})}{(12-6\sqrt{2})}\cdot\frac{(12+6\sqrt{2})}{(12+6\sqrt{2})} = \frac{-72-36\sqrt{2}+72\sqrt{2}+72}{144-72}$$

$$= \frac{36\sqrt{2}}{72} = \frac{\sqrt{2}}{2}$$

Hence $t_1 = 12-6\sqrt{2}$

$$t_2 = (12-6\sqrt{2})\left(\frac{\sqrt{2}}{2}\right) = 6\sqrt{2}-6$$

$$t_3 = (12-6\sqrt{2})\left(\frac{\sqrt{2}}{2}\right)^2 = (12-6\sqrt{2})\left(\frac{1}{2}\right)$$

$$= 6-3\sqrt{2}$$

That is, $12-6\sqrt{2},\ -6+6\sqrt{2},\ 6-3\sqrt{2}$

THE BINOMIAL THEOREM

Consider the following array:

$$(x+y)^0 = 1$$
$$(x+y)^1 = 1x + 1y$$
$$(x+y)^2 = 1x^2 + 2xy + 1y^2$$
$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3$$
$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$$
$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

We can continue using the procedure for multiplying polynomials to extend this sequence of equations as far as we choose. Thus:

$$(x+y)^6 = (x+y)(x+y)^5$$
$$= 1x^6 + 5x^5y + 10x^4y^2 + 10x^3y^3 + 5x^2y^4$$
$$+ xy^5 + 1x^5y + 5x^4y^2 + 10x^3y^3$$
$$+ 10x^2y^4 + 5xy^5 + 1y^6$$
$$= 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4$$
$$+ 6xy^5 + 1y^6$$

Proceeding in this way we can obtain an expression (expansion) for each higher power of the binomial $(x+y)$ in a step-by-step fashion.

However,
we would like a general formula for the expansion of $(x+y)^n$ when $n \in \mathbb{N}_0$ which would enable us to expand $(x+y)^7$, for example, without having to obtain all of the preceding expansions.

Moreover, we would like a general formula for each term in this expansion. With this end in view, we study the preceding expansions and make the following observations:

1. The number of terms in the expansion of $(x+y)^n$ is $n+1$.

2. The coefficient of the first term in the expansion is 1.

3. The exponent for x is n in the first term and decreases by one in each succeeding term; the exponent for y is 0 in the first term and increases by one in each succeeding term. (The sum of the exponents for x and y in any term is n.)

4. The coefficient for any term after the first is the product of the coefficient of the preceding term and the exponent of x for that term divided by the number of that term.

Using these observations we readily obtain:

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

We generalize the preceding discussion with the following theorem, called the binomial theorem:

THEOREM 6-14.

If $x, y \in \mathbb{C}$, $n, r \in \mathbb{N}_0$, $xy \neq 0$, and $n \geq r$, then

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{1 \cdot 2}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^{n-3}y^3 + \cdots + \frac{\overbrace{n(n-1)(n-2)\cdots(n-r+1)}^{r \text{ factors}}}{\underbrace{1 \cdot 2 \cdot 3 \cdots r}_{r \text{ factors}}}x^{n-r}y^r + \cdots + y^n$$

The conclusion of the binomial theorem is often called the binomial formula.

EXAMPLE 1

Expand $(2a-b)^8$

SOLUTION

The required expansion may be obtained by substituting $2a$ for x and $-b$ for y in the binomial formula. We have:

$$(2a-b)^8 = [2a + (-b)]^8$$

$$= (2a)^8 + 8(2a)^7(-b) + \frac{8 \cdot 7}{1 \cdot 2}(2a)^6(-b)^2$$

$$+ \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}(2a)^5(-b)^3 + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4}(2a)^4(-b)^4$$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(2a)^3(-b)^5$$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}(2a)^2(-b)^6$$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}(2a)(-b)^7$$

$$+ \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}(-b)^8$$

$= 256a^8 - 1024a^7b + 1792a^6b^2 - 1792a^5b^3$
$+ 1120a^4b^4 - 448a^3b^5 + 112a^2b^6$
$- 16ab^7 + b^8$

In the preceding example, we observe the indicated product $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ in the numerator of the term involving $(-b)^8$. Products such as this appear so frequently in mathematics that a special notation has been introduced for them, namely, "$n!$".

Thus $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$n!$ is read "n factorial"

$(n+1)(n!) = (n+1)!$

This equation suggests a way to define $0!$. Let $n = 0$ and we have $1! = 1 \cdot 0!$

Therefore, we accept the following definition: $0! = 1$

EXAMPLE 2

Evaluate the following expressions

(a) $\frac{800!}{798!}$ (b) $\frac{7!}{3!\,4!}$ (c) $\frac{3!+5!}{(7-4)!}$

(d) $\frac{(n+2)!}{n!}$ (e) $\frac{(n-r+1)!}{(n-r-1)!}$

SOLUTIONS

(a) $\frac{800!}{798!} = \frac{800 \cdot 799 \cdot (798!)}{798!}$

$= 800 \cdot 799 = 639{,}200$

(b) $\frac{7!}{3!\,4!} = \frac{7 \cdot 6 \cdot 5 \cdot (4!)}{3 \cdot 2 \cdot 1\ (4!)} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

(c) $\frac{3!+5!}{(7-4)!} = \frac{3!\,(1+5 \cdot 4)}{3!} = 21$

(d) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)\,n!}{n!}$

$= (n+2)(n+1)$

(e) $$\frac{(n-r+1)!}{(n-r-1)!} = \frac{(n-r+1)(n-r)[(n-r-1)!]}{(n-r-1)!}$$

$$= (n-r+1)(n-r)$$

A EXERCISES

1. Using the binomial theorem, write the expansion of each of the following:

(a) $(a+b)^{10} = a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10ab^9 + b^{10}$

(b) $(x+3)^7 = x^7 + 7x^6(3) + 21x^5(3)^2 + 35x^4(3)^3 + 35x^3(3)^4 + 21x^2(3)^5 + 7x\cdot(3)^6 + 3^7$

$= x^7 + 21x^6 + 189x^5 + 945x^4 + 2835x^3 + 5103x^2 + 5103x + 2187$

(c) $(x-5)^8 = x^8 + 8x^7(-5) + 28x^6(-5)^2 + 56x^5(-5)^3 + 70x^4(-5)^4 + 56x^3(-5)^5 + 28x^2(-5)^6 + 8x(-5)^7 + (-5)^8$

$= x^8 - 40x^7 + 700x^6 - 7000x^5 + 43750x^4 - 175000x^3 + 437500x^2 - 625000x + 390{,}625$

(d) $(3a+2b)^5 = (3a)^5 + 5(3a)^4(2b) + 10(3a)^3(2b)^2$
$+ 10(3a)^2(2b)^3 + 5(3a)(2b)^4 + (2b)^5$
$= 243a^5 + 810a^4b + 1080a^3b^2 + 720a^2b^3$
$+ 240ab^4 + 32b^5$

(e) $(b-c)^6 = b^6 + 6b^5(-c) + 15b^4(-c)^2$
$+ 20b^3(-c)^3 + 15b^2(-c)^4$
$+ 6b(-c)^5 + (-c)^6$
$= b^6 - 6b^5c + 15b^4c^2 - 20b^3c^3$
$+ 15b^2c^4 - 6bc^5 + c^6$

(f) $(2a-3b)^5 = (2a)^5 + 5(2a)^4(-3b)$
$+ 10(2a)^3(-3b)^2 + 10(2a)^2(-3b)^3$
$+ 5(2a)(-3b)^4 + (-3b)^5$

$= 32a^5 - 240a^4b + 720a^3b^2$
$- 1080a^2b^3 + 810ab^4 - 243b^5$

(2) Evaluate each of the following:

(a) $\frac{19!}{16!} = \frac{19 \cdot 18 \cdot 17 \cdot (16!)}{16!} = 19 \cdot 18 \cdot 17$

$= 5814$

(b) $\frac{52!}{49!\,3!} = \frac{52 \cdot 51 \cdot 50 \cdot (49!)}{3 \cdot 2 \cdot 1\,(49!)} = \frac{52}{2} \cdot \frac{51}{3} \cdot 50$

$= 26 \cdot 17 \cdot 50 = 22100$

(c) $\frac{1}{3!} + \frac{1}{5!} = \frac{1}{3!} + \frac{1}{3!\,(5 \cdot 4)}$

$= \frac{20+1}{5!} = \frac{21}{5!} = \frac{3 \cdot 7}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{7}{40}$

(d) $\frac{(4+3)!}{4!\,3!} = \frac{7 \cdot 6 \cdot 5 \cdot (4!)}{3 \cdot 2 \cdot 1 \cdot (4!)} = 35$

(e) $\frac{15! - 13!}{14!} = \frac{13!\,(15 \cdot 14 - 1)}{14 \cdot (13!)} = \frac{15 \cdot 14 - 1}{14}$

$= 15 - \frac{1}{14} = \frac{209}{14}$

(f) $\frac{3!\,4!}{4! - 3!} = \frac{3!\,4!}{3!\,(4-1)} = \frac{4 \cdot 3 \cdot 2}{3} = 8$

③ Simplify:

(a) $$\frac{(n+1)\,n\,(n-1)!}{(n+1)!} = \frac{(n+1)!}{(n+1)!} = 1$$

(b) $$\frac{n!}{(n-2)!} = \frac{n\,(n-1)(n-2)!}{(n-2)!} = n^2-n$$

(c) $$\frac{(n+3)!}{(n+2)!} = \frac{(n+3)(n+2)!}{(n+2)!} = n+3$$

(d) $$\frac{(n-2)!}{(n+1)!} = \frac{(n-2)!}{(n+1)(n)(n-1)(n-2)!}$$

$$= \frac{1}{(n+1)(n)(n-1)} = \frac{1}{n^3-n}$$

(e) $$\frac{(n!)^2}{(n+1)!\,(n-1)!} = \frac{(n!)(n!)}{(n+1)(n!)(n-1)!}$$

$$= \frac{n\,(n-1)!}{(n+1)(n-1)!} = \frac{n}{n+1}$$

(f) $(n+1)[(n+1)!+n!]$

$= (n+1)\,n!\,[(n+1)+1] = (n+2)(n+1)\,n!$

$= (n+2)!$

(g) $\dfrac{1}{(n-1)!} - \dfrac{1}{n!} = \dfrac{1}{(n-1)!} - \dfrac{1}{n(n-1)!}$

$= \dfrac{n-1}{n!}$

(h) $\dfrac{(n+3)! + n!(n+2)}{n!(n+2)!} = \dfrac{n!\left[(n+3)(n+2)(n+1) + (n+2)\right]}{n!(n+2)!}$

$= \dfrac{(n+2)\left[(n+3)(n+1) + 1\right]}{(n+2)(n+1)!} = \dfrac{n^2+4n+4}{(n+1)!}$

$= \dfrac{(n+2)^2}{(n+1)!}$

4. Solve for n.

(a) $n! = 720 \longleftrightarrow n = 6 \longleftrightarrow \{6\}$

(b) $\dfrac{n!}{(n-2)!} = 72 \longleftrightarrow \dfrac{n(n-1)(n-2)!}{(n-2)!} = 72$

$n^2 - n = 72 \longleftrightarrow n^2 - n - 72 = 0$

$\longleftrightarrow (n-9)(n+8) = 0 \longleftrightarrow n = 9 \vee n = -8$ (reject)

$\therefore \{9\}$

Check: $\dfrac{9!}{7!} = 9 \cdot 8 = 72$

(c) $n! = 930\,(n-2)!$

$$n(n-1)(n-2)! = 930\,(n-2)!$$

$$n^2 - n = 930 \leftrightarrow n^2 - n - 930 = 0$$

$$930 = 2\cdot 3\cdot 5\cdot 31 = 30\cdot 31$$

$$\therefore\ (n-31)(n+30) = 0 \qquad \{31\}$$

Check: $31! = 930\,(29!)$

$$\frac{31!}{29!} = 930 \leftrightarrow \frac{31\cdot 30\cdot(29!)}{(29!)} = 930$$

(d) $$\frac{(2n+1)!\,(2n-1)!}{[(2n)!]^2} = \frac{11}{10}$$

$$\leftrightarrow \frac{(2n+1)[(2n)!][(2n-1)!]}{[(2n)!][(2n)(2n-1)!]} = \frac{11}{10}$$

$$\leftrightarrow \frac{2n+1}{2n} = \frac{11}{10} \leftrightarrow 10(2n+1) = 22n$$

$$\leftrightarrow 20n + 10 = 22n \leftrightarrow 10 = 2n \leftrightarrow n = 5$$

$$\therefore\ \{5\}$$

(5) Prove the identity $\frac{1}{(n-1)!}+\frac{1}{n!}=\frac{n+1}{n!}$

$$\frac{1}{(n-1)!}+\frac{1}{n(n-1)!}=\frac{n+1}{n(n-1)!}=\frac{n+1}{n!}$$

MORE ABOUT THE BINOMIAL THEOREM

Referring to Theorem 6-14, we observe that the term involving y^r is the $(r+1)^{th}$ term of the expansion of $(x+y)^n$.

A If $x, y \in C$, $r, n \in N_0$, $xy \neq 0$, and $n \geq r$, then the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is

$$\frac{n(n-1)\cdots(n-r+1)}{r!}x^{n-r}y^r.$$

☆ Note that this formula gives all terms except the first. The first term is, of course, x^n.

The fifth term of the expansion of $(x+y)^8$ can be obtained by substituting 8 for n and 4 for r in statement **A**.

Thus the 5^{th} term in the expansion of $(x+y)^8$ is $\frac{8\cdot7\cdot6\cdot5}{4!}x^{8-4}y^4$,

or $70x^4y^4$.

The fraction $\frac{8\cdot7\cdot6\cdot5}{4!}$ can be expressed more compactly by multiplying both numerator and denominator by $4\cdot3\cdot2\cdot1$, or $4!$.

We have:

$$\frac{8\cdot7\cdot6\cdot5}{4!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4!\,4!}$$

$$= \frac{8!}{4!\,4!}$$

We can use the same procedure to write the coefficient

$$\frac{n(n-1)\cdots(n-r+1)}{r!}$$ in statement $\boxed{A}$

in a more compact form.

This time we multiply the numerator and denominator by $(n-r)!$.

We have:

$$\frac{n(n-1)\cdots(n-r+1)}{r!} =$$

$$\frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots 3\cdot 2\cdot 1}{r!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!}.$$

Accordingly, the coefficient of the term involving y^r in the expansion of $(x+y)^n$ is $\frac{n!}{r!\,(n-r)!}$.

We will find it convenient to use the symbol ${}_nC_r$, or $\binom{n}{r}$, to represent this coefficient.

Thus ${}_8C_3 = \binom{8}{3} = \frac{8!}{3!\,5!}$ is the coeffecient of x^5y^3 in the expansion of $(x+y)^8$.

$_{11}C_8 = \binom{11}{8} = \frac{11!}{8!3!}$ is the coeffecient of x^3y^8 in the expansion of $(x+y)^{11}$; and $_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$ is the coeffecent of $x^{n-r}y^r$ in the expansion of $(x+y)^n$.

Using this notation we can write the binomial formula in the following form:

B

$$\begin{aligned}(x+y)^n = &\binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y \\ &+ \binom{n}{2}x^{n-2}y^2 + \cdots \\ &+ \binom{n}{r}x^{n-r}y^r + \cdots \\ &+ \binom{n}{n}x^{n-n}y^n\end{aligned}$$

provided $n \in N$, and $xy \neq 0$

Using the summation (sigma Σ) notation, we may write the binomial theorem as follows:

C

If $x, y \in \mathbb{C}$, $n, r \in \mathbb{N}_1$, $xy \neq 0$, and $n \geq r$, then $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$

We also observe that $\boxed{A}$ can now be written somewhat more compactly as follows:

D

If $x, y \in \mathbb{C}$, $n, r \in \mathbb{N}_1$, $xy \neq 0$, and $n \geq r$, the the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is $\binom{n}{r} x^{n-r} y^r$.

EXAMPLE 3

Use statement C to expand $(2a-b)^8$.

SOLUTION

Since statement C is merely another way of stating the binomial theorem, we again substitute $2a$ for x, $(-b)$ for y, and 8 for n.

Thus $(2a-b)^8 = \sum_{r=0}^{8} \binom{8}{r}(2a)^{8-r}(-b)^r$.

Expanding the right member of this equation and simplifying each term, we obtain the result shown in example 1, page 113.

As early as 1996 or 1998, with the introduction of computer algebra systems such as Derive into the TI-92, we could either expand((2a−b)^8)

or Σ(nCr(8,r)*(2*a)^(8−r)*(−b)^r, r, 0, 8)

yields: $256a^8 - 1024\cdot a^7b + 1792a^6b^2 - 1792a^5b^3 + 1120a^4b^4 - 448a^3b^5 + 112a^2b^6 - 16ab^7 + b^8$

EXAMPLE 4

(a) Find the eleventh term in the expansion of $(2a-1)^{13}$.

(b) Find the term involving b^3 in the expansion of $(a-2b^{1/2})^9$.

SOLUTIONS

(a) Let $n=13$, $r=10$, $x=2a$, and $y=-1$ in statement D. The eleventh term of $(2a-1)^{13}$

$$= \binom{13}{10}(2a)^3(-1)^{10} = \frac{13!}{10!\,3!}\,8a^3(1) = \frac{13\cdot 12\cdot 11}{3\cdot 2\cdot 1}(8a^3)$$

$$= (4a^3)(13\cdot 11\cdot 4) = 2288\,a^3$$

(b) The required term involves $(-2b^{1/2})^6$. Therefore $r=6$, $n=9$, $x=a$, $y=-2b^{1/2}$ in statement D. The required term is

$$\binom{9}{6}a^3(-2b^{1/2})^6 = \frac{9\cdot 8\cdot 7}{3\cdot 2\cdot 1}\,a^3\cdot 64b^3$$

$$= 3\cdot 8\cdot 7\cdot 32\,a^3\cdot b^3 = 5376\,a^3b^3$$

The coefficients in the expansions of $(x+y)^n$ shown on page 109 form the following array called Pascal's triangle.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

If the "1" at the top is called row 0, and the first item in each row is called item 0, then item j in row n can be found from the formula:

$$\binom{n}{j} = \frac{n!}{(n-j)!\,j!}$$

Also, row n of the triangle gives the coefficients of the expansion of $(a+b)^n$.

What is $\binom{0}{0}$? $\binom{0}{0} = \frac{0!}{(0-0)!\,0!} = 1$

We can write the triangle as:

$\binom{1}{0} = 1$, $\binom{1}{1} = 1$, $\binom{2}{0} = 1$, $\binom{n}{0} = 1$, $\binom{r}{r} = 1$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

etc ...

$$\binom{n}{0} \quad \binom{n}{1} \quad \binom{n}{2} \quad \cdots\cdots \binom{n}{r}\binom{n}{r+1}$$

Notice that $\binom{2}{1} + \binom{2}{2} = \binom{3}{2}$

and $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$

and, in general, $\boxed{E}$ $\boxed{\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}}$

Since $\binom{n}{r} = \left(\frac{n!}{r!(n-r)!}\right)$, E can be written

as: $\boxed{F}$

$$\boxed{\frac{(n+1)!}{r!(n-r+1)!} = \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}}$$

It can be readily shown that F is an identity and we have:

THEOREM 7-14.

If $n, r \in C$, $r \leq n$, and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$,

then $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

This theorem is known as Pascal's theorem for binomial coefficients.

If we expand the coefficients in formula B (p. 124), we may write the binomial expansion in the following form:

$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \ldots$$

If we let $x = 1$ in this formula, we obtain:

G

$$(1+y)^n = 1 + ny + \frac{n(n-1)}{2!}y^2 + \frac{n(n-1)(n-2)}{3!}y^3 + \ldots$$

In more advanced courses in mathematics it is proved that there are conditions under which formula G is valid even when n is not a counting number (N_1).

For example, G is valid for any rational number n when $|y| < 1$. In fact, if $n \in Q - N_0$, $y \in R$, and $|y| < 1$, the right member of G is a convergent infinite series whose "sum" is the left member of G.

EXAMPLE 5

Expand $\frac{1}{1-x}$ by using statement G.

SOLUTION $\frac{1}{1-x} = (1-x)^{-1}$

We substitute $-x$ for y and -1 for n.

$$(1-x)^{-1} = [1+(-x)]^{-1}$$

$$= 1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \ldots$$

$$\therefore (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

If $|x| < 1$, the right member of this equation, $1 + x + x^2 + x^3 + \ldots$, is an infinite geometric series whose sum is $\frac{1}{1-x}$ according to Theorem 5-14 (page 87)

$$S_n = \frac{a - ar^n}{1-r} \longrightarrow \lim_{n \to \infty} S_n = \frac{a}{1-r} \text{ when } |r| < 1.$$

It is also noteworthy that the expansion for $(1-x)^{-1}$ can be obtained as follows: $\frac{1}{1-x} = 1 + x + x^2 + \ldots$

$$\begin{array}{r|l}
 & 1 + x + x^2 + x^3 + \ldots \\
\hline
1-x & 1 + 0x + 0x^2 + 0x^3 + 0x^4 + \ldots \\
 & \underline{1 - x} \\
 & \quad x \\
 & \quad \underline{x - x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{x^2 - x^3} \\
 & \qquad\quad x^3 \\
 & \qquad\quad \underline{x^3 - x^4}
\end{array}$$

EXAMPLE 6

Use statement G to find $\sqrt{10}$ to the nearest thousandth.

SOLUTION

$$\sqrt{10} = (9+1)^{1/2} = \left[9\left(1+\frac{1}{9}\right)\right]^{1/2}$$

$$= 9^{1/2}\left(1+\frac{1}{9}\right)^{1/2} = 3\left(1+\frac{1}{9}\right)^{1/2}$$

Concentrating on $\left(1+\frac{1}{9}\right)^{1/2}$, using statement G, $y=\frac{1}{9}$ and $n=\frac{1}{2}$.

$$\left(1+\frac{1}{9}\right)^{\frac{1}{2}} \approx 1 + \left(\frac{1}{2}\right)\left(\frac{1}{9}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{1}{9}\right)^2$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{1}{9}\right)^3 + \ldots$$

$$\approx 1 + \frac{1}{18} - \frac{\frac{1}{4}}{2}\frac{1}{9^2} + \frac{\frac{3}{8}}{6}\cdot\frac{1}{9^3} + \ldots$$

$$\approx 1 + \frac{1}{18} - \frac{1}{8}\cdot\frac{1}{81} + \frac{1}{16}\cdot\frac{1}{9^3} + \ldots$$

$$\approx 1 + \frac{1}{18} - \frac{1}{648} + \frac{1}{11664} - \ldots$$

$$\approx 1 + 0.0555 - 0.0015 + 0.0001$$

$$\approx 1.0541 \quad \therefore \sqrt{10} = 3\left(1+\frac{1}{9}\right)^{1/2} \approx 3.162$$

Observe that in order to get the answer correct to 3 decimal places, we retained 4 decimal places until the final step.

(c) $(2.01)^8 = \left(2 + \frac{1}{100}\right)^8$

$$= 2^8 + 8(2)^7\left(\tfrac{1}{100}\right) + 28(2)^6\left(\tfrac{1}{100}\right)^2$$

$$+ 56(2)^5\left(\tfrac{1}{100}\right)^3 + 70(2)^4\left(\tfrac{1}{100}\right)^4$$

$$+ 56(2)^3\left(\tfrac{1}{100}\right)^5 + 28(2)^2\left(\tfrac{1}{100}\right)^6$$

$$+ 8(2)\left(\tfrac{1}{100}\right)^7 + \left(\tfrac{1}{100}\right)^8$$

$$\doteq 256 + 10.24 + 0.1792 + 0.001792$$

$$+ 0.0000112 + \dots \approx 266.421$$

(d) $\sqrt{5}$ See example on page 133.

$$\sqrt{5} = 5^{\frac{1}{2}} = (4+1)^{\frac{1}{2}} = \left[4\left(1+\tfrac{1}{4}\right)\right]^{\frac{1}{2}}$$

$$= 4^{\frac{1}{2}}\left(1+\tfrac{1}{4}\right)^{1/2} = 2\left(1+\tfrac{1}{4}\right)^{1/2}$$

Concentrating now on $\left(1+\frac{1}{4}\right)^{1/2}$, using statement G from bottom of page 130, with $x=1$ and $y=\frac{1}{4}$, we obtain:

$$\left(1+\tfrac{1}{4}\right)^{1/2} = 1 + \frac{1}{2}\cdot\frac{1}{4} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{4}\right)^2$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(\frac{1}{4}\right)^3 \dots$$

$$\doteq 1 + \frac{1}{8} - \frac{1}{128} + \frac{3}{2944}$$

$$\doteq 1.118 \quad \therefore \sqrt{5} \doteq 2.236$$

(e) $\sqrt{17} = 17^{1/2} = (16+1)^{1/2} = \left[16\left(1+\frac{1}{16}\right)\right]^{1/2}$

$$= 16^{1/2}\left(1+\frac{1}{16}\right)^{1/2} = 4\left(1+\frac{1}{16}\right)^{1/2}$$

$$\left(1+\frac{1}{16}\right)^{1/2} = 1 + \frac{1}{2}\cdot\frac{1}{16} + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{16}\right)^2$$

$$+ \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(\frac{1}{16}\right)^3 - \cdots$$

$$\doteq 1 + 0.03125 - 0.000488\ldots + \underbrace{0.0000000\,22\ldots}_{\text{inconsequential}}$$

$$\doteq 1.030762$$

$$4(1.0308) = 4.1232$$

$$\therefore \sqrt{17} \approx 4.123$$

(f) $\sqrt[3]{28} = 28^{\frac{1}{3}} = (27+1)^{\frac{1}{3}} = \left[27\left(1+\frac{1}{27}\right)\right]^{1/3}$

$$= 27^{1/3}\left(1+\frac{1}{27}\right)^{1/3} = 3\left(1+\frac{1}{27}\right)^{\frac{1}{3}}$$

$$\left(1+\frac{1}{27}\right)^{\frac{1}{3}} \approx 1 + \frac{1}{3}\cdot\frac{1}{27} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(\frac{1}{27}\right)^2$$

$$+ \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3\cdot 2}\left(\frac{1}{27}\right)^3 + \cdots$$

$$\approx 1 + 0.0123456\ldots - 0.000152$$

$$\approx 1.0121936 \approx 1.0122$$

$$\therefore \sqrt[3]{28} \doteq 3(1.0122) = 3.0366 \doteq 3.037$$